

Synchronization of Inhibitory Pulse-Coupled Oscillators in Delayed Random and Line Networks

(Invited Paper)

Johannes Klinglmayr¹ and Christian Bettstetter^{1,2}

¹ University of Klagenfurt, Mobile Systems Group, Institute of Networked and Embedded Systems, Austria

² Lakeside Labs GmbH, Klagenfurt, Austria

Abstract—In a system of pulse-coupled oscillators with delays, inhibitory coupling assures synchronization and convergence if the oscillators are fully connected. This paper investigates inhibitory coupling in non-fully-connected networks by means of simulation. First, we show that an existing inhibitory coupling scheme does not perform well in random networks. Second, we propose a simple modification, which tends to improve the synchronizing behavior of inhibitory coupling.

Index Terms—Synchronization, inhibitory coupling, meshed networks, pulse-coupled oscillator, self-organization;

I. INTRODUCTION

The synchronous flashing of fireflies is a fascinating example for self-organization in nature (see [1], [2]). Every firefly emits a light pulse with a certain frequency. Whenever recording a pulse, a firefly slightly adjusts itself to the registered shifted frequency. This principle yields favorable properties, such as scalability, simplicity and decentralization. These make firefly synchronization an interesting approach for synchronization in communication systems (see [3], [4]).

A mathematical model for firefly synchronization is the theory of pulse-coupled oscillators as used in [5]. The entities of the system (the fireflies) are represented as oscillators and interact via infinitely short pulse signals. Each oscillator has a phase ϕ that evolves over time from 0 to 1 and resets afterwards. Whenever an oscillator reaches the threshold 1 (firing event) it emits a pulse, which is received by all other elements in the system. Each receiving oscillator adjusts, triggered by an incoming pulse, by performing a “phase jump”, which is a change in an oscillator’s phase position.

Synchronization can be proven for idealized environments [5], where all oscillators are fully connected and all actions are performed without any delay. For use in technical systems, however, nonnegligible restrictions have to be incorporated. Examples include delays (see [4], [6], [7]) and inability to send infinitesimally short signals (see [7], [8]). Modifications to the synchronization method presented in [5] are needed, and the proof for convergence can no longer be applied.

Our paper [9] suggests a firefly-based synchronization method that takes into account time-varying delays (as they might occur in technical systems). Based on “inhibitory coupling” (negative phase jumps), it assures synchronization in delay-free systems and convergence to a close-to-synchrony state in delayed systems. Both types of convergence were proven for fully-connected systems of oscillators. This paper

extends our investigations to non-fully-connected systems. In particular, we analyze the synchronization behavior of inhibitory coupling in random networks and line networks. Simulation results show that the algorithm presented in [9] does not achieve proper synchronization in such cases. Therefore we present a slight modification of the inhibitory-coupled oscillator algorithm. This modification has positive effects in non-fully-connected networks and can reassure synchronization in some cases.

II. PULSE-COUPLED OSCILLATORS

Synchronization of pulse-coupled oscillators (PCOs) is intensively studied in many disciplines, in particular in physics and dynamic systems theory. This section gives a brief introduction as needed for understanding this paper.

A. Delay-Free Systems

Mirollo and Strogatz [5] use PCOs to model and prove firefly synchronization in delay-free systems. An oscillator’s phase rate is modeled to be constant, in particular $\frac{d\phi}{dt} = 1$ with $\phi \in [0, 1]$ and periodic boundary conditions. Whenever $\phi = 1$ an oscillator emits a pulse. Whenever an oscillator receives a pulse it immediately adjusts its phase from $\phi(t)$ to $\phi(t^+)$, where t^+ denotes an infinitesimally short time instant after t . The new phase of the adjusting oscillator is given by

$$\phi(t^+) = \min(H(\phi(t)), 1), \quad (1)$$

where $H(\cdot)$, the transfer function, is given by

$$H(\phi) = U^{-1}(U(\phi) + \varepsilon). \quad (2)$$

The function $U(\cdot)$ has to fulfill certain conditions, such as being twice continuously differentiable and $U(0) = 0$, $U(1) = 1$, $U' > 0$, and $U'' < 0$.

The system applies *excitatory coupling* if phase jumps are positive ($\varepsilon > 0$, as in [5]). It performs *inhibitory coupling* if phase jumps are negative ($\varepsilon < 0$), motivated from neuroscience modeling. With excitatory coupling, a convergence proof for delay-free systems and almost all initial conditions was elaborated [5].

B. Delayed Systems

If delays are present in a system of oscillators, the model presented above no longer works, but the phase adjustment as shown in (1) must be modified [10]. We define τ_{ij} as the time

it takes for the signal to travel from oscillator i to a receiving oscillator j . To overcome possible echoing effects, a refractory phase ϕ_{ref} is used [10]. Within this phase, no phase adjustments are performed. If oscillator i fires, the phase adjustment is as follows: $\phi_i(t) = 1 \Rightarrow$ for i : $\phi_i(t^+) = 0$ and for $j \neq i$:

$$\begin{cases} \phi_j(t + \tau_{ij}^+) = \phi_j(t + \tau_{ij}) & \phi_j(t + \tau_{ij}) \leq \phi_{\text{ref}} \\ \phi_j(t + \tau_{ij}^+) = \min(H(\phi_j(t + \tau_{ij})), 1) & \phi_j(t + \tau_{ij}) > \phi_{\text{ref}} \end{cases} \quad (3)$$

A fully synchronous state is no longer feasible. Hence with delayed coupling only a *close-to-synchrony* state can be achieved. Without a refractory phase within a system with delays several clusters of oscillators may emerge and synchronization not necessarily exists [11].

Excitatory coupling was applied for applications to technical systems, but its synchronizing behavior could not be proven for large networks (see [4], [6], [10]). Inhibitory coupling was investigated mainly in neuroscience (see [12], [13]), where positive synchronization behavior was revealed but a guarantee for synchronization for all initial conditions could not be drawn. Our paper [9] presented a slight modification of the inhibitory coupling (*inhibitory coupling with self-adjustment*) and gave a proof for synchronization.

III. INHIBITORY COUPLING WITH SELF-ADJUSTMENT

The synchronization algorithm under investigation as introduced in [9] works as follows. The phase of an oscillator increases linearly over time, i.e., we have $\frac{d\phi}{dt} = 1$ at all non-event times, with $\phi \in [0, 1]$ and periodic boundary conditions. An oscillator emits a pulse whenever $\phi = 1$ is reached from below. Whenever an oscillator i emits a pulse, it immediately *self-adjusts* from $\phi_i(t)$ to $\phi_i(t^+)$. Upon reception of such a pulse from another oscillator j at time $t + \tau_{ij}$, an adjustment is performed from $\phi_j(t + \tau_{ij})$ to $\phi_j(t + \tau_{ij}^+)$. The overall procedure after reaching the firing threshold 1 is $\phi_i(t) = 1 \Rightarrow$

$$\begin{cases} \phi_i(t^+) = H(\phi_i(t)) \\ \phi_j(t + \tau_{ij}^+) = \phi_j(t + \tau_{ij}) & j \neq i \wedge \phi_j(t + \tau_{ij}) \leq \phi_{\text{ref}} \\ \phi_j(t + \tau_{ij}^+) = H(\phi_j(t + \tau_{ij})) & j \neq i \wedge \phi_j(t + \tau_{ij}) > \phi_{\text{ref}} \end{cases} \quad (4)$$

Here the transfer function is defined as

$$H(\phi) := (1 + \alpha) \cdot \phi \quad (5)$$

where $\alpha \in (-1, 0)$. The rules that an oscillator has to follow are summarized in Algorithm 1. The refractory phase is chosen as in [9].

Section IV will show that the performance behavior of this algorithm in non-fully-connected networks is weak due to the possibility of phase locks. One way to overcome this disadvantage is to break the stiff network influences and coupling by an approach as given in Algorithm 2, where oscillators that hit the firing threshold always self-adjust but send a fire pulse only with probability p_{send} .

IV. SYNCHRONIZATION ANALYSIS

We compare the synchronization performance of the excitatory coupling scheme (as shown in (3)) and both inhibitory

Algorithm 1 Inhibitory coupling with self-adjustment in a delayed system.

- 1) Each PCO has its own phase ϕ from 0 to 1.
 - 2) When $\phi(t) = 1$, the PCO fires and adjusts its phase to $\phi(t^+) = \phi(t) + \Delta\phi(t)$ (instantaneous self-coupling).
 - 3) When receiving a firing at time t' , the PCO adjusts its phase to $\phi(t'^+) = \phi(t') + \Delta\phi(t')$.
 - 4) For $\Delta\phi(t')$ with $\alpha < 0$:
 - a) If $\phi(t') > 1 - |\alpha| + 2\tau_{\text{max}}$, set $\Delta\phi(t') = \alpha \cdot \phi(t')$.
 - b) If $\phi(t') \leq 1 - |\alpha| + 2\tau_{\text{max}}$, set $\Delta\phi(t') = 0$.
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Algorithm 2 Inhibitory coupling with self-adjustment in a delayed system with probabilistic fires.

- 1) Each PCO has its own phase ϕ from 0 to 1.
 - 2) Whenever $\phi(t) = 1$, the PCO adjusts its phase to $\phi(t^+) = \phi(t) + \Delta\phi(t)$ (instantaneous self-coupling), and sends a pulse with probability p_{send} .
 - 3) When receiving a firing at time t' , the PCO adjusts its phase to $\phi(t'^+) = \phi(t') + \Delta\phi(t')$.
 - 4) For $\Delta\phi(t')$ with $\alpha < 0$:
 - a) If $\phi(t') > 1 - |\alpha| + 2\tau_{\text{max}}$, set $\Delta\phi(t') = \alpha \cdot \phi(t')$.
 - b) If $\phi(t') \leq 1 - |\alpha| + 2\tau_{\text{max}}$, set $\Delta\phi(t') = 0$.
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coupling schemes (Algorithms 1 and 2). All schemes use the transfer function (5) with parameter $\alpha_{\text{exc}} = -\alpha_{\text{inh}}$.

A. Networks

The set of nodes is called Ω . We investigate the synchronization performance for three different network topologies: Erdős-Rényi random graphs, dynamic random graphs, and a line graph. They are constructed as follows:

- *Erdős-Rényi random graph*. For given Ω , each possible edge is inserted with probability p . The resulting graph is not necessarily connected, but the restriction on connected graphs is no necessary precondition [14].
- *Dynamic random graph*. For given Ω , for any time instance an edge out of this set is present with probability p . Hence the dynamic random graph represents a different Erdős-Rényi random graph at any point in time. For a dynamic random graph the topology changes constantly and therefore also the connectivity of the network can change unremittingly.
- *Line graph*. The nodes are numbered ascendingly with $n = |\Omega|$, and for any node $i \in \{2, \dots, n-1\}$ there is an edge to node $i-1$ and $i+1$. Node 1 and n have an edge to 2 and $n-1$ respectively. The line graph is considered to be a worst-case scenario for synchronization since the maximum number of a node's edges is at its minimum.

B. Synchrony Metric

To measure the phase offset and distance between the oscillators we use the following metric

$$\Pi(\phi) := \frac{\max_{i,j \in \Omega} [\min(|\phi_i(t) - \phi_j(t)|, \omega - |\phi_i(t) - \phi_j(t)|)]}{\omega}, \quad (6)$$

where $\phi := \{\phi_i\}_{i \in \Omega}$, and ω is the cycle length. For the inhibitory coupling with self-adjustment $\omega = |\alpha|$, for the excitatory coupling as seen in (3) we have $\omega = 1$. This metric ensures a fair comparison in terms of phase difference, since all offsets are measured to their relative cycle length.

C. Algorithm Parameters and Simulation Methodology

The refractory period for excitatory coupling is set to $\phi_{\text{ref}} = 0.081$. The pulse delay is set as a continuously changing variable with values within $[0, 5]$ percent of the cycle length. The coupling strength is $\alpha = -0.99$. The network order is $|\Omega| = 10$. Due to the delay, a fully synchronous state is not feasible; therefore we say that a steady state or close-to-synchrony state is achieved for a phase constellation with $\Pi(t) \leq 0.03$.

The synchronization algorithms are simulated on specific networks. The decision whether the system reached the close-to-synchrony condition or not is made after 25 cycles. For random networks, 1000 simulations are performed for every edge probability. Results show the fraction of networks reaching close-to-synchrony.

D. Synchronization in Erdős-Rényi Random Networks

Inhibitory coupling according to Algorithm 1 leads to synchrony in fully-connected networks [9]. We now investigate its performance in an Erdős-Rényi random network and compare it to the performance of excitatory coupling.

Figure 1 shows the fraction of networks reaching the close-to-synchrony state as a function of the edge probability. The performance of both excitatory and inhibitory coupling is shown. We observe two completely different behaviors. Using excitatory coupling, the system reaches the close-to-synchrony state almost surely as soon as the edge probability is above 0.5. So excitatory coupling manages to converge in non-fully-connected networks if the edge probability is high enough. Inhibitory coupling, however, shows a low fraction of convergence for all edge probabilities lower than 1. It shows a convex convergence tendency, which only reaches synchrony when fully connected. Excitatory coupling on the other hand performs a strong convergence increase with low edge probability and achieves synchrony from then on. Inhibitory coupling reaches 100% synchronization only with 100% edge probability. This shows that the current synchronization proof as in [9] cannot be generalized for arbitrary networks.

E. Synchronization in Dynamic Random Networks

Within dynamic random networks, edges and connectivity change continuously. In Figure 2 we see the close-to-synchrony fraction depending on the edge probability for both

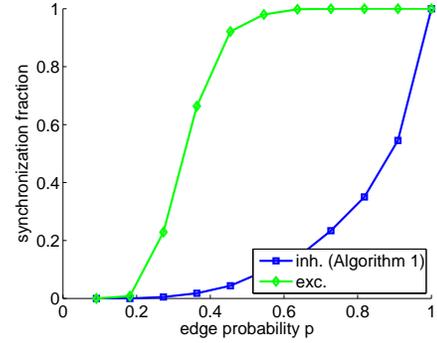


Fig. 1. Synchronization behavior with Erdős-Rényi random networks.

excitatory and inhibitory coupled systems. Here the performance picture changes. Inhibitory coupling outperforms excitatory coupling, and both coupling schemes converge faster than in Figure 1. The fraction of being close-to-synchrony rises very quickly, especially compared to the Erdős-Rényi random networks regarding inhibitory coupling. With an edge probability of $p_{\text{send}} = 0.3$ already above 99% of all networks are in the close-to-synchrony state. The chance of an edge between nodes, as provided within a dynamic random network, is a beneficial factor for synchronization with inhibitory coupling; excitatory coupling is less influenced by these dynamic edge connections.

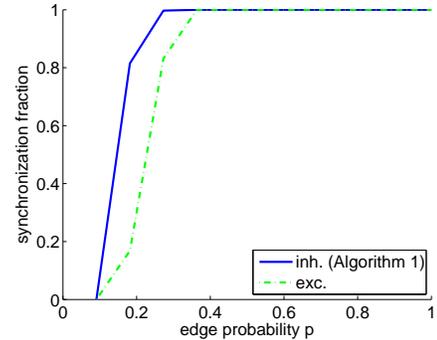


Fig. 2. Synchronization behavior with dynamic random networks.

F. Synchronization with Probabilistic Firings

The performance of Algorithm 1 shown in Figure 2 can be interpreted as follows: It is not necessary for the inhibitory coupling with self-adjustment to send fires at the end of every cycle. It seems to be sufficient to send with a lower firing frequency. Therefore we introduce Algorithm 2. It only differs from Algorithm 1 by the parameter p_{send} , which reduces the chances of firing at the end of a cycle.

Let us now study the case of a line topology, since it is one of the worst case topologies and reveals some aspects that partly explain the performance difference between Figures 1 and 2. Within a line topology phase lock situations may appear as is visualized in Table I. Since, with inhibitory coupling, a

receiving node is adjusting its phase toward 0, instead of 1 as in excitatory coupling, it is not possible to forward a firing signal to another node; this deteriorates phase communication. By introducing some randomness to the algorithm as seen in Algorithm 2, the chances increase that phase lock situations are resolved.

TABLE I
A PHASE LOCK SITUATION WITH 3 OSCILLATORS ON A LINE TOPOLOGY.

	ϕ_1	ϕ_2	ϕ_3
t_0 :	0.8	0.4	0.5
t_1 :	1	0.6	0.7
t_1^+ :	0.5	0.3	0.7
t_2 :	0.8	0.6	1
t_2^+ :	0.8	0.3	0.5
t_3 :	1	0.5	0.7
t_3^+ :	0.5	0.25	0.7
t_4 :	0.8	0.55	1
t_4^+ :	0.8	0.275	0.5

We see the phase evolution of three delay-free oscillators with $H(\cdot)$ as in (5) and $\alpha = -0.5$. With every firing event ϕ_2 is decreased, and thus will never fire itself. Hence no information can be transported from ϕ_1 to ϕ_3 or vice versa, and the two oscillators are phase locked. Since ϕ_1 and ϕ_3 will never decrease their phase difference synchronization of the full system is not possible.

We demonstrate the influence of the probabilistic coupling with the line topology in Figure 3. The plot shows the percentage of initial conditions that synchronize with a given fire probability $p_{\text{send}} \in [0, 1]$ and inhibitory coupling. The case of Algorithm 1 is represented with $p_{\text{send}} = 1$ and shows that in general certain fires are not beneficial for this topology. Overall we see a very poor synchronization behavior with inhibitory coupling, at most around 8% of initial conditions synchronize. Excitatory coupling with $p_{\text{send}} = 1$ on the other hand yields 100% successful synchronization rate. Nevertheless, even in this worst case scenario, probabilistic coupling has advantageous effects especially for the parameter values of $p_{\text{send}} \in [0.5, 0.7]$. In Figure 4 the mean synchronization level using different initial conditions versus p_{send} is plotted. Also the standard deviation is shown. In general, the reached synchronization level is far from being synchronized, but drastically improves with probabilistic firings.

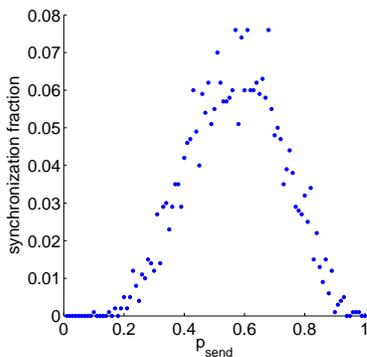


Fig. 3. Fraction of networks that synchronize with a line topology using Algorithm 2.

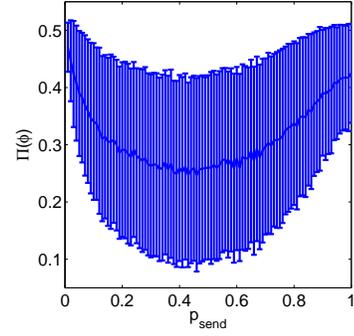


Fig. 4. Mean synchrony metric value of networks with a line topology using Algorithm 2.

Applying Algorithm 2 to Erdős-Rényi random networks we see further positive properties of this approach. In Figure 5, basically the setting from Figure 2 is repeated, but now supplemented by the influence of Algorithm 2. We see a tremendous increase of networks that achieve the close-to-synchrony state. Furthermore, an appropriate choice of the parameter p_{send} can be found within the value 0.5. As seen in the line topology the probabilistic firings break phase lock situations and thus improve the synchronization behavior. The performance of excitatory coupling can almost be reached with $p_{\text{send}} = 0.5$. Especially with high edge probability synchronization convergence is almost as good as with excitatory coupling.

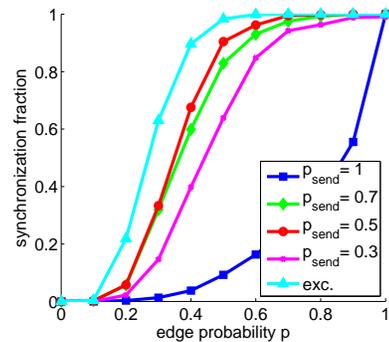


Fig. 5. Fraction of networks that synchronize within an Erdős-Rényi network topology using Algorithm 2 and different p_{send} .

V. CONCLUSIONS

In this paper we investigated the influence of the network topology on inhibitory coupling with self-adjustment. Whereas, as proven in [9], a close-to-synchrony state is achieved in a fully connected network, this is no more true if we rely on Erdős-Rényi random networks with edge probability smaller than one. On the contrary, for dynamic networks simulations show that the close-to-synchrony state can still be achieved even with low edge probability (around 0.3). Modifying the algorithm from [9] slightly, by randomly reducing the number of firing events we, however, can exploit

the positive effect of the dynamic networks to a wider range and could, by simulations, ensure that within Erdős-Rényi random networks with a high edge probability (above 0.7) the close-to-synchrony state is achieved.

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